The topology of random lemniscates

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A random lemniscate of high degree

\[ \Gamma = \left\{ z \in \mathbb{C} : \left| \frac{p(z)}{q(z)} \right| = 1 \right\} \quad \text{(plotted on the Riemann sphere)} \]
The Erdős Lemniscate Problem (1958): Find the maximal planar length of a monic polynomial lemniscate of degree \( n \).

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\Lambda := \{ z \in \mathbb{C} : |p(z)| = 1 \}.
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- conjectured extremal is \( p(z) = z^n - 1 \) (the “Erdős lemniscate”)
- confirmed locally and asymptotically (Fryntov, Nazarov, 2008)

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Probabilistic perspective on the Erdös lemniscate problem

Sample \( p \) from the Kac ensemble.

Q. What is the average length of \(|\Lambda|\)?

Answer (L., Ramachandran): The average length approaches a constant,

\[
\lim_{n \to \infty} \mathbb{E}|\Lambda| = C \approx 8.3882.
\]

Corollary: “The Erdös lemniscate is an outlier.”
Randomize the rational lemniscate

\[ \Gamma = \left\{ z \in \mathbb{C} : \left| \frac{p(z)}{q(z)} \right| = 1 \right\} \]

by randomizing the coefficients of \( p \) and \( q \):

\[ p(z) = \sum_{k=0}^{n} a_k z^k, \quad \text{and} \quad q(z) = \sum_{k=0}^{n} b_k z^k, \]

where \( a_k \) and \( b_k \) are independent complex Gaussians:

\[ a_k \sim \mathcal{N}_\mathbb{C} \left( 0, \binom{n}{k} \right), \quad b_k \sim \mathcal{N}_\mathbb{C} \left( 0, \binom{n}{k} \right). \]
Random samples plotted on the Riemann sphere

Degree $n = 100, 200, 300, 400, 500$.
Rational lemniscates: three guises

- **Complex Analysis**: pre-image of unit circle under rational map

\[ \left| \frac{p(z)}{q(z)} \right| = 1 \]

- **Potential Theory**: logarithmic equipotential (for point-charges)

\[ \log |p(z)| - \log |q(z)| = 0 \]

- **Algebraic Geometry**: special real-algebraic curve

\[ p(x + iy)p(x + iy) - q(x + iy)q(x + iy) = 0 \]
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Prevalence of lemniscates (pure and applied)

- **Approximation theory**: Hilbert’s lemniscate theorem
- **Conformal mapping**: Bell representation of $n$-connected domains
- **Holomorphic dynamics**: Mandelbrot and Julia lemniscates
- **Numerical analysis**: Arnoldi lemniscates
- **2-D shapes**: proper lemniscates and conformal welding
- **Harmonic mapping**: critical sets of complex harmonic polynomials
- **Gravitational lensing**: critical sets of lensing maps
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The lensing potential:

\[ \kappa |z|^2 - \sum_{i=1}^{n} m_i \log |z - z_i| \]

The lensing map (gradient of potential):

\[ z \mapsto \kappa z - \sum_{i=1}^{n} \frac{m_i}{\bar{z} - \bar{z}_i} . \]

The critical set (vanishing of the Jacobian) of this map is a rational lemniscate:

\[ \left\{ z \in \mathbb{C} : \left| \sum_{i=1}^{n} \frac{m_i}{(z - z_i)^2} \right| = \kappa \right\} . \]

The caustic: Image of the critical lemniscate under the lensing map.
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Rational lemniscates in gravitational lensing

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Petters and Witt (1996) observed a transition to no cusps while tuning $\kappa$.

How many cusps are on a random caustic?
Programmatic problem: Study Hilbert’s sixteenth problem (on the topology of real algebraic manifolds) from the random viewpoint.

“The random curve is 4% Harnack.”
-P. Sarnak, 2011

Several recent studies address this problem (Nazarov, Sodin, Gayet, Welschinger, Sarnak, Wigman, Canzani, Beffara, Fyodorov, Lerario, L.).

The crux: the desired features (connected components, arrangements) are highly non-local.
A. Eremenko and W. Hayman (1999) posed and solved a spherical version of the Erdös lemniscate problem:

**Spherical Lemniscate Problem:** Find the maximal spherical length of a rational lemniscate of degree $n$.

**Answer:** The maximum is exactly $2 \pi n$.

**Q. What is the average length of $|\Gamma|$?**

**Answer (Lerario, L., 2016)**

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$$\mathbb{E}|\Gamma| = \frac{\pi^2}{2} \sqrt{n}.$$
Integral geometry formula for length of $\Gamma$:

$$\frac{|\Gamma|}{\pi} = \int_{SO(3)} |\Gamma \cap gS^1| \, dg$$

Taking expectation on both sides:

$$E|\Gamma| = \pi \int_{SO(3)} E|\Gamma \cap gS^1| \, dg.$$

Rotational invariance $\implies \, dg$ integrand is constant. Thus, the average length

$$E|\Gamma| = \pi E|\Gamma \cap S^1|$$

reduces to a one-dimensional Kac-Rice problem.
Hilbert’s sixteenth problem for real algebraic curves: Study the number of connected components and classify the possible arrangements of components.

Specialized to lemniscates, this problem has a complete solution:

For a rational lemniscate of degree $n$ the number of components is at most $n$, and any arrangement of up to $n$ components can occur.
Hilbert’s sixteenth problem restricted to lemniscates

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Non-local statistics

- connected components
- arrangements of components (nesting)
- long components (giant components?)
- Morsifications (topological equivalence classes of landscapes)
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The average number of connected components

\[ N(\Gamma) - \text{number of connected components} \]

\[ c \cdot n \leq \mathbb{E}N(\Gamma) \leq \left( \frac{32 - \sqrt{2}}{28} \right) n + O(\sqrt{n}), \]

The upper bound is based on the average number of meridian tangents.

The lower bound uses the barrier method.
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The average number of connected components increases linearly in $n$, that is, there exist constants $c_1, c_2 > 0$ such that

$$c_1 n \leq \mathbb{E} b_0(\Gamma) \leq c_2 n.$$ 

The proof uses an adaptation of the “barrier method” introduced by F. Nazarov and M. Sodin (2007): localize the problem and establish a positive probability (independent of $n$) of finding a component inside a disk of radius $n^{-1/2}$. 
Thermodynamic limit: meromorphic lemniscates

Heuristic: rescaling by $1/\sqrt{n}$ and letting $n \to \infty$ leads to the lemniscate

$$\left\{ z \in \mathbb{C} : \left| \frac{f(z)}{g(z)} \right| = 1 \right\}$$

determined by the (translation invariant) ratio of two GAFs:

$$f(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{\sqrt{k!}}, \quad a_k \sim N_C(0, 1) \quad \text{i.i.d.},$$

and $g$ is an independent copy of $f$. 
Prevalence of arrangements in the nesting graph

The barrier method can also be used to study the probability of any fixed arrangement occurring on the scale $1/\sqrt{n}$.

Given any arrangement $A$, for every open disk $D$ of radius $n^{-1/2}$ in the Riemann sphere there is a positive probability (independent of $n$) that $\Gamma \cap D$ realizes the arrangement $A$. 
Morsifications of $|r(z)|$

**Problem:** Study the whole family of level curves

$$\Gamma_t = \{ z \in \mathbb{C} : |r(z)| = t \}$$

and the arrangement of singular levels.
Thank you!
Hilbert’s lemniscate theorem: Polynomial lemniscates are dense in the space of closed Jordan curves.

Given a closed Jordan curve \( G \) and \( \varepsilon > 0 \) there exists a lemniscate \( \Gamma \) that contains \( G \) in its interior with \( \text{dist}(z, G) < \varepsilon \) for each \( z \in \Gamma \).
Lemniscates in potential theory

For polynomial lemniscate domains \( \{ z : |p(z)| > 1 \} \) the function

\[
\frac{1}{n} \log |p(z)|
\]

is the harmonic Green’s function with pole at infinity.
Bell representation of multiply-connected domains:

Every non-degenerate $n$-connected planar domain is conformally equivalent to some lemniscate domain of the form:

$$\left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{|z - b_k|} \right| < r \right\}.$$  

(such domains have algebraic Bergman and Szegö kernels)
Lemniscates in holomorphic dynamics

(Mandelbrot lemniscates.)
Lemniscates in numerical analysis

Arnoldi lemniscates: Iteration scheme used for approximating the largest eigenvalue of a large matrix.
D. Mumford and E. Sharon proposed a conformal welding procedure to “fingerprint” 2-dimensional shapes using diffeomorphisms of the circle.

If the shape is assumed to be a lemniscate the corresponding fingerprint is the $n$th root of a finite Blaschke product (P. Ebenfelt, D. Khavinson, and H.S. Shapiro).
The polynomial planar harmonic mapping

\[ z \mapsto p(z) + \overline{q(z)} \]

has a critical set that is a rational lemniscate:

\[ \left\{ z \in \mathbb{C} : \left| \frac{p'(z)}{q'(z)} \right| = 1 \right\}. \]

So-called “lensing maps” (from gravitational lensing theory)

\[ z \mapsto \bar{z} - \sum_{i=1}^{n} \frac{m_i}{z - z_i}, \]

also have critical sets that are rational lemniscates:

\[ \left\{ z \in \mathbb{C} : \left| \sum_{i=1}^{n} \frac{m_i}{(z - z_i)^2} \right| = 1 \right\}. \]
Lemniscates in classical Mathematics

The arclength of Bernoulli’s lemniscate

\[ \{ |z^2 - 1| = 1 \} \]

is a famous elliptic integral (of the first kind):

\[
2\sqrt{2} \int_{0}^{1} \frac{1}{\sqrt{1 - x^4}} \, dx \approx 7.416
\]

The same integral shows up in classical statics (length of an elastica) and mechanics (period of a pendulum).