The topology of random lemniscates

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joint work (Proc. London Math. Soc., 2016) with Antonio Lerario
and joint work (in preparation) with Koushik Ramachandran

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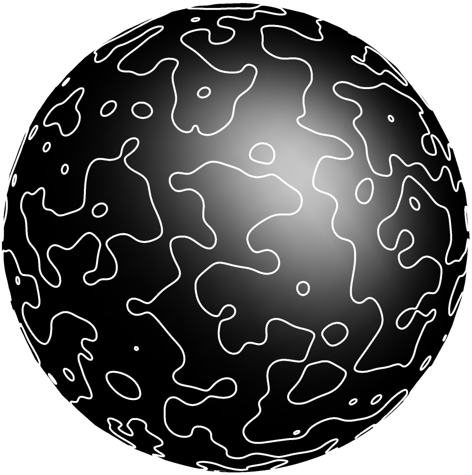
A random lemniscate of high degree



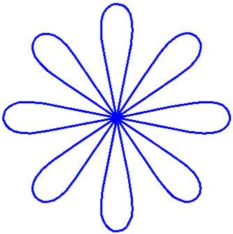
$$\Gamma = \left\{ z \in \mathbb{C} : \left| \frac{p(z)}{q(z)} \right| = 1 \right\}$$ (plotted on the Riemann sphere)

Probabilistic perspective on the Erdös lemniscate problem

The Erdös Lemniscate Problem (1958): Find the maximal planar length of a monic polynomial lemniscate of degree n.

$$\Lambda := \{z \in \mathbb{C} : |p(z)| = 1\}.$$

- conjectured extremal is $p(z) = z^n - 1$ (the "Erdös lemniscate")
- confirmed locally and asymptotically (Fryntov, Nazarov, 2008)



From the probabilistic viewpoint:

Q. What is the average length of Λ?

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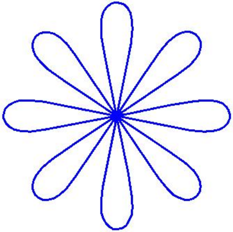

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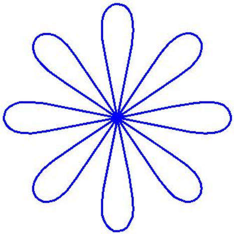

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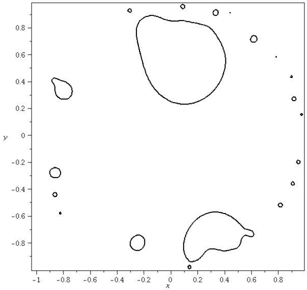


Sample p from the Kac ensemble.
Q. What is the average length of $|\Lambda|$?

Answer (L., Ramachandran): The average length approaches a constant,

$$\lim_{n\to\infty} \mathbb{E}|\Lambda| = C \approx 8.3882.$$

Corollary: "The Erdös lemniscate is an outlier."

Random rational lemniscates: choosing the ensemble

Randomize the rational lemniscate

$$\Gamma = \left\{ z \in \mathbb{C} : \left| \frac{p(z)}{q(z)} \right| = 1 \right\}$$

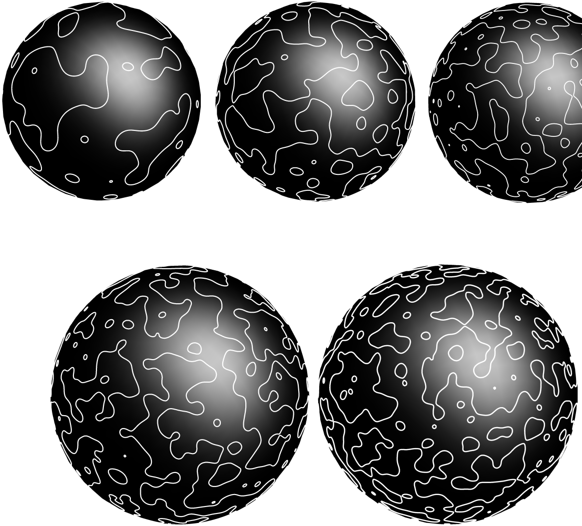
by randomizing the coefficients of p and q:

$$p(z) = \sum_{k=0}^{n} a_k z^k, \quad \text{and} \quad q(z) = \sum_{k=0}^{n} b_k z^k,$$

where a_k and b_k are independent complex Gaussians:

$$a_k \sim N_{\mathbb{C}}\left(0, \binom{n}{k}\right), \quad b_k \sim N_{\mathbb{C}}\left(0, \binom{n}{k}\right).$$

Random samples plotted on the Riemann sphere



Degree $n = 100, 200, 300, 400, 500$.

Rational lemniscates: three guises

- **Complex Analysis:** pre-image of unit circle under rational map

$$\left|\frac{p(z)}{q(z)}\right| = 1$$

- **Potential Theory:** logarithmic equipotential (for point-charges)

$$\log|p(z)| - \log|q(z)| = 0$$

- **Algebraic Geometry:** special real-algebraic curve

$$p(x+iy)\overline{p(x+iy)} - q(x+iy)\overline{q(x+iy)} = 0$$

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Prevalence of lemniscates (pure and applied)

- **Approximation theory:** Hilbert's lemniscate theorem
- **Conformal mapping:** Bell representation of n-connected domains
- **Holomorphic dynamics:** Mandelbrot and Julia lemniscates
- **Numerical analysis:** Arnoldi lemniscates
- **2-D shapes:** proper lemniscates and conformal welding
- **Harmonic mapping:** critical sets of complex harmonic polynomials
- **Gravitational lensing:** critical sets of lensing maps

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Rational lemniscates in gravitational lensing

The lensing potential:

$$\kappa|z|^2 - \sum_{i=1}^{n} m_i \log|z - z_i|$$

The lensing map (gradient of potential):

$$z \mapsto \kappa z - \sum_{i=1}^{n} \frac{m_i}{\bar{z} - \bar{z}_i}.$$

The critical set (vanishing of the Jacobian) of this map is a rational lemniscate:

$$\left\{ z \in \mathbb{C} : \left| \sum_{i=1}^{n} \frac{m_i}{(z - z_i)^2} \right| = \kappa \right\}.$$

The caustic: Image of the critical lemniscate under the lensing map.

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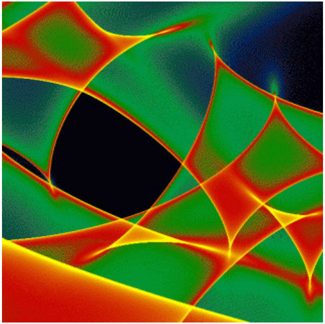
Caustics and cusps



Petters and Witt (1996) observed a transition to no cusps while tuning κ.

How many cusps are on a random caustic?

Non-local topology of random real algebraic manifolds

Programmatic problem: Study Hilbert's sixteenth problem (on the topology of real algebraic manifolds) from the random viewpoint.

"The random curve is 4% Harnack."
-P. Sarnak, 2011

Several recent studies address this problem (Nazarov, Sodin, Gayet, Welschinger, Sarnak, Wigman, Canzani, Beffara, Fyodorov, Lerario, L.).

The crux: the desired features (connected components, arrangements) are highly *non-local*.

The spherical length of rational lemniscates

A. Eremenko and W. Hayman (1999) posed and solved a spherical version of the Erdös lemniscate problem:

Spherical Lemniscate Problem: Find the maximal spherical length of a rational lemniscate of degree n.

Answer: The maximum is exactly $2\pi n$.

Q. What is the average length of $|\Gamma|$?

Answer (Lerario, L., 2016)

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Average length and integral geometry

Integral geometry formula for length of Γ:

$$\frac{|\Gamma|}{\pi} = \int_{SO(3)} |\Gamma \cap gS^1| dg$$

Taking expectation on both sides:

$$\mathbb{E}|\Gamma| = \pi \int_{SO(3)} \mathbb{E}|\Gamma \cap gS^1| dg.$$

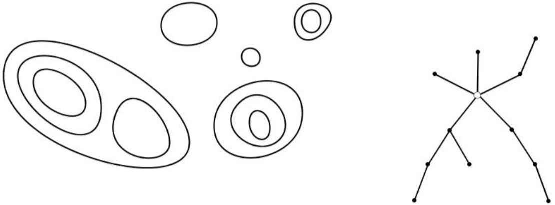
Rotational invariance $\implies$ dg integrand is constant.
Thus, the average length

$$\mathbb{E}|\Gamma| = \pi \mathbb{E}|\Gamma \cap S^1|$$

reduces to a one-dimensional Kac-Rice problem.

Hilbert's sixteenth problem restricted to lemniscates

Hilbert's sixteenth problem for real algebraic curves: Study the number of connected components and classify the possible arrangements of components.



Specialized to lemniscates, this problem has a complete solution:

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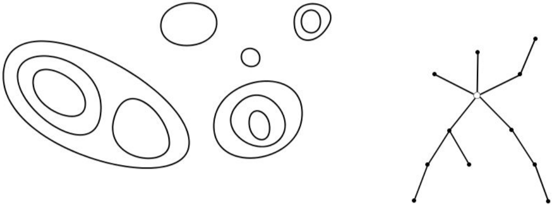

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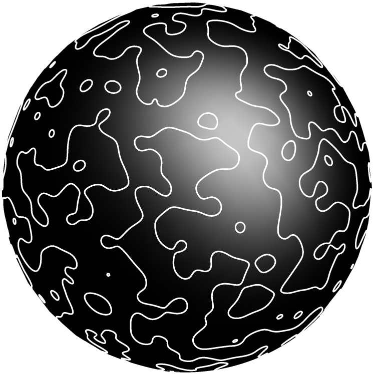
Non-local statistics

- connected components
- arrangements of components (nesting)
- long components (giant components?)
- Morsifications (topological equivalence classes of landscapes)



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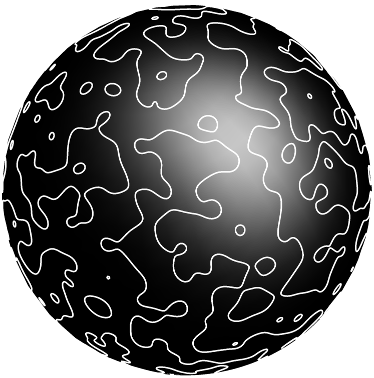

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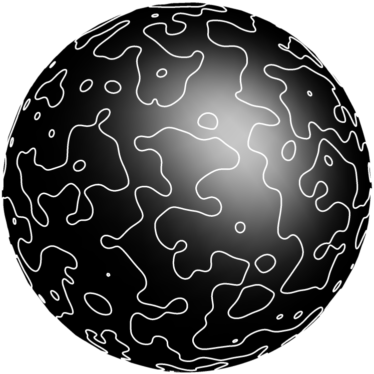

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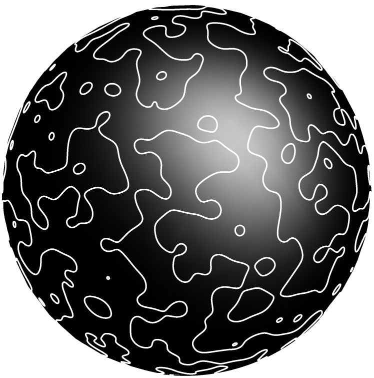

The average number of connected components

$N(\Gamma)$ - number of connected components

$$c \cdot n \leq \mathbb{E}N(\Gamma) \leq \left(\frac{32-\sqrt{2}}{28}\right) n + O(\sqrt{n}),$$

The upper bound is based on the average number of meridian tangents.

The lower bound uses the barrier method.

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The average number of connected components

The average number of connected components increases linearly in n, that is, there exist constants $c_1, c_2 > 0$ such that

$$c_1 n \leq \mathbb{E} b_0(\Gamma) \leq c_2 n.$$

The proof uses an adaptation of the "barrier method" introduced by F. Nazarov and M. Sodin (2007): localize the problem and establish a positive probability (independent of n) of finding a component inside a disk of radius $n^{-1/2}$.

Thermodynamic limit: meromorphic lemniscates

Heuristic: rescaling by $1/\sqrt{n}$ and letting $n \to \infty$ leads to the lemniscate

$$\left\{ z \in \mathbb{C} : \left| \frac{f(z)}{g(z)} \right| = 1 \right\}$$

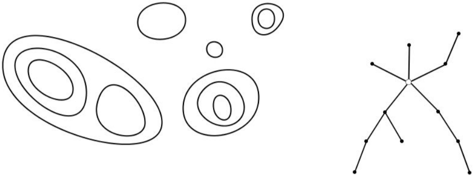
determined by the (translation invariant) ratio of two GAFs:

$$f(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{\sqrt{k!}}, \quad a_k \sim N_{\mathbb{C}}(0,1) \quad i.i.d.,$$

and g is an independent copy of f.

Prevalence of arrangements in the nesting graph

The barrier method can also be used to study the probability of any fixed arrangement occurring on the scale $1/\sqrt{n}$.



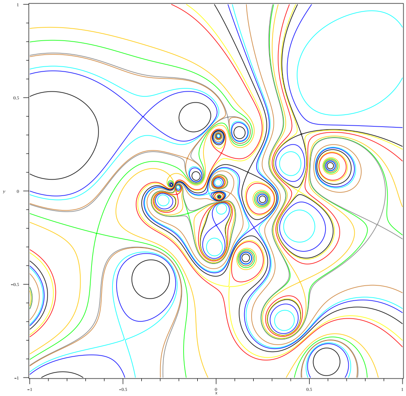
Given any arrangement A, for every open disk D of radius $n^{-1/2}$ in the Riemann sphere there is a positive probability (independent of n) that $\Gamma \cap D$ realizes the arrangement A.

Morsifications of $|r(z)|$

Problem: Study the whole family of level curves

$$\Gamma_t = \{z \in \mathbb{C} : |r(z)| = t\}$$

and the arrangement of singular levels.



Thank you!

Lemniscates in approximation theory

Hilbert's lemniscate theorem: Polynomial lemniscates are dense in the space of closed Jordan curves.

Given a closed Jordan curve G and $\varepsilon > 0$ there exists a lemniscate Γ that contains G in its interior with $\text{dist}(z, G) < \varepsilon$ for each $z \in \Gamma$.

Lemniscates in potential theory

For polynomial lemniscate domains $\{z : |p(z)| > 1\}$ the function

$$\frac{1}{n} \log |p(z)|$$

is the harmonic Green's function with pole at infinity.

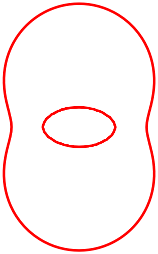
Lemniscates in conformal geometry

Bell representation of multiply-connected domains:

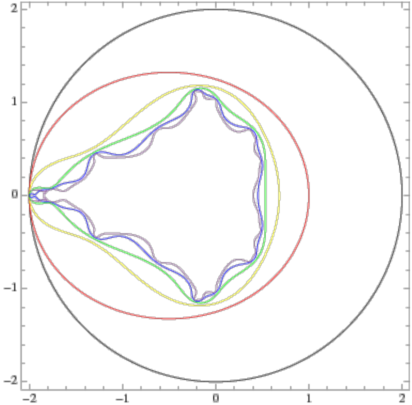
Every non-degenerate n-connected planar domain is conformally equivalent to some lemniscate domain of the form:

$$\left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < r \right\}.$$

(such domains have algebraic Bergman and Szegö kernels)

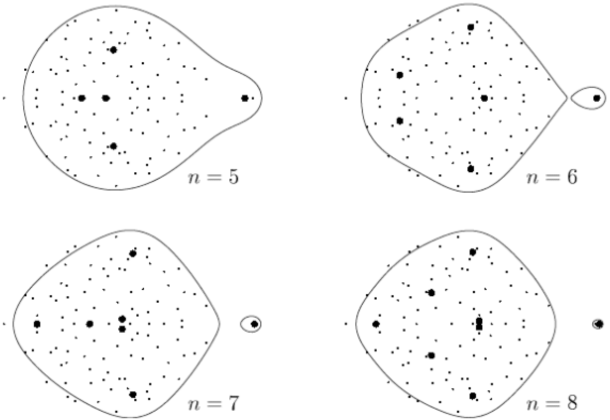


Lemniscates in holomorphic dynamics



(Mandelbrot lemniscates.)

Lemniscates in numerical analysis



Arnoldi lemniscates: Iteration scheme used for approximating the largest eigenvalue of a large matrix.

Lemniscates in 2-D shape compression

D. Mumford and E. Sharon proposed a conformal welding procedure to "fingerprint" 2-dimensional shapes using diffeomorphisms of the circle.

If the shape is assumed to be a lemniscate the corresponding fingerprint is the nth root of a finite Blaschke product (P. Ebenfelt, D. Khavinson, and H.S. Shapiro).

Lemniscates as critical sets of harmonic mappings

The polynomial planar harmonic mapping

$$z \mapsto p(z) + \overline{q(z)}$$

has a critical set that is a rational lemnsicate:

$$\left\{ z \in \mathbb{C} : \left| \frac{p'(z)}{q'(z)} \right| = 1 \right\}.$$

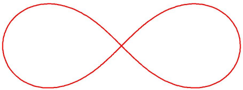
So-called "lensing maps" (from gravitational lensing theory)

$$z \mapsto \bar{z} - \sum_{i=1}^{n} \frac{m_i}{z - z_i},$$

also have critical sets that are rational lemniscates:

$$\left\{ z \in \mathbb{C} : \left| \sum_{i=1}^{n} \frac{m_i}{(z - z_i)^2} \right| = 1 \right\}.$$

Lemniscates in classical Mathematics



The arclength of Bernoulli's lemniscate

$$\left\{ \left| z^2 - 1 \right| = 1 \right\}$$

is a famous elliptic integral (of the first kind):

$$2\sqrt{2} \int_0^1 \frac{1}{\sqrt{1 - x^4}} dx \approx 7.416$$

The same integral shows up in classical statics (length of an elastica) and mechanics (period of a pendulum).